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Multioptimum of a Convex Functional

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1. INTRODUCTION

Let U, V be a pair of convex sets in a normed linear space X. The points $\bar{u} \in U$, $\bar{v} \in V$ are called proximal if

$$\|\bar{u} - \bar{v}\| = d(U, V) = \inf_{u \in U, v \in V} \|u - v\|.$$

It is easily observed that if the points $\bar{u} \in U$, $\bar{v} \in V$ are proximal, then they are mutually nearest to each other from the respective sets. However, the converse implication is generally not true, even for Chebyshev[1] sets U, V. In this connection it is convenient to restate here the following result from Pai [9]: "In order that for each pair U, V of convex sets in X, points $\bar{u} \in U$, $\bar{v} \in V$ that are mutually nearest to each other be proximal, it is necessary and sufficient that the space X be smooth." In the present paper this result is embedded in the answers to the following general questions pertaining to convex optimization in locally convex spaces.

QUESTION 1. Let f be a convex functional defined on a Hausdorff locally convex linear topological space X. Let U, V be a pair of convex sets in X. A pair $(\bar{u}, \bar{v}) \in U \times V$ is called a *multioptimum* for f if

$$f(\bar{u} - \bar{v}) = \inf_{v \in V} f(\bar{u} - v) = \inf_{u \in U} f(u - \bar{v}), \tag{1.1}$$

and it is called simply an *optimum* for f if $\bar{u} - \bar{v}$ is an *optimum* for f on $U - V$, i.e.,

$$f(\bar{u} - \bar{v}) = \inf_{u \in U, v \in V} f(u - v). \tag{1.2}$$

[1] See [12, p. 103] for the definition.

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Then we ask: Under what conditions is a *multioptimum* $(\bar{u}, \bar{v})$ an *optimum* for f? More generally, we are concerned with

QUESTION 2. Let X_i, $i = 1, 2,..., n$, be Hausdorff locally convex linear topological spaces and let $K_i \subset X_i$, $i = 1, 2,..., n$, be convex sets. Let F be a convex functional defined on $\prod_{i=1}^{n} X_i$. Denote by $F_{\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_{i+1},...,\bar{x}_n}$, $i = 1, 2,..., n$, the convex functionals defined on X_i by

$$F_{\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_{i+1},...,\bar{x}_n}(x_i) = F(\bar{x}_1, \bar{x}_2,..., \bar{x}_{i-1}, x_i, \bar{x}_{i+1},..., \bar{x}_n). \tag{1.3}$$

We call $(\bar{x}_1, \bar{x}_2,..., \bar{x}_n) \in \prod_{i=1}^{n} K_i$ a *multioptimum* for F if

$$F(\bar{x}_1,..., \bar{x}_n) = \inf_{x \in K_i} F_{\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_{i+1},...,\bar{x}_n}(x_i), \qquad i = 1, 2,..., n, \tag{1.4}$$

and simply an *optimum* for F if

$$F(\bar{x}_1,..., \bar{x}_n) = \inf_{\substack{x_i \in K_i \\ i=1,2,...,n}} F(x_1, x_2,..., x_n). \tag{1.5}$$

Then we ask: Under what conditions is a *multioptimum* $(\bar{x}_1,..., \bar{x}_n)$ an *optimum* for F?

Question 2, of course, contains Question 1 as a special case upon taking $F(u, v) = f(u - v)$. However, as it turns out, for particular cases such as the case when f is a gauge function, necessary as well as sufficient conditions can be given in order that, for each pair U, V of convex sets, $(\bar{u}, \bar{v}) \in U \times V$ being a *multioptimum* for f imply that it is an *optimum* for f.

The main results pertaining to Question 2 are given in Section 2. In Section 3 we are concerned with Question 1 and also deal there with a special case when f is given as a certain seminorm. Section 4 deals with the important special cases when the convex sets K_i of Section 2 and the convex sets U, V of Section 3 are contained in subspaces of finite dimension. In Section 5 we discuss two applications: (1) multivariate constrained convex optimization, and (2) global simultaneous approximation.

We take the standard framework of convex analysis as adopted in [8] or [11] and recall here those notions that will frequently be used in the sequel. Let X, Y be complex linear spaces in duality, $<, >$ denoting the duality relation. For topologies on X and Y, we take topologies compatible with the given duality $<, >$. Equipped with these, X, Y become Hausdorff locally convex linear topological spaces. We say $f \in \text{conv}(X)$ if $f: X \to \mathbb{R} \cup \{+\infty\}$ is proper, i.e., $f \not\equiv +\infty$ and it is convex. Let χ_K stand for the indicator function

$$\begin{aligned} \chi_K(x) &= 0, \qquad x \in K, \\ &= \infty, \qquad x \notin K, \end{aligned}$$

and let $\text{dom}(f) = \{x \in X / f(x) < \infty\}$. Following Laurent [6, p. 335], we say that a function $f \in \text{conv}(X)$ is d-continuous if f is continuous on int-dom(f). The subdifferential of f at $\bar{x}$ is $\partial f(\bar{x}) = \{y \in Y / f(x) \geqslant f(\bar{x}) + \text{Re}\langle x - \bar{x}, y\rangle, \forall x \in X\}$. The following result of Moreau and Rockafellar (cf. Holmes [3, p. 25]) will frequently be employed. Let $f_1, f_2 \in \text{conv}(X)$. Suppose there exists some point in $\text{dom}(f_1) \cap \text{dom}(f_2)$ at which one of the two functions is continuous. Then for each $\bar{x} \in X$ one has $\partial(f_1 + f_2)(\bar{x}) = \partial f_1(\bar{x}) + \partial f_2(\bar{x})$.

2. Characterization of Optimum and Multioptimum in Question 2

Let the linear spaces X_i and Y_i be in duality, $\langle , \rangle_i$ denoting the duality relation between them, $i = 1, 2,..., n$. For the product spaces $\prod_{i=1}^n X_i$ and $\prod_{i=1}^n Y_i$ we take the following duality that corresponds to the given dualities between X_i and Y_i:

$$\langle (x_1, ..., x_n), (y_1, ..., y_n)\rangle = \langle x_1, y_1\rangle_1 + \langle x_2, y_2\rangle_2 + \cdots + \langle x_n, y_n\rangle_n .$$

THEOREM 2.1. *Let* $F \in \text{conv}(\prod_{i=1}^n X_i)$ *and let* $K_i \subset X_i$ *be convex,* $i = 1, 2,..., n$. *Assume that either*

(H_1) $\text{dom}(F) \cap \prod_{i=1}^n \text{int}(K_i) \neq \varnothing$

or

(H_1') *F is* d-*continuous and* $\text{int-dom}(F) \cap \prod_{i=1}^n K_i \neq \varnothing$

holds. Then $(\bar{x}_1, ..., \bar{x}_n) \in \prod_{i=1}^n K_i$ *is an optimum for F if and only if there exists* $(y_1, y_2, ..., y_n) \in \prod_{i=1}^n Y_i$ *such that*

(i) $(y_1, y_2, ..., y_n) \in \partial F(\bar{x}_1, ..., \bar{x}_n)$ *and*

(ii) $\text{Re}\langle \bar{x}_i, y_i\rangle_i = \min_{x_i \in K_i} \text{Re}\langle x_i, y_i\rangle_i$.

Proof. We observe that $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) \in \prod_{i=1}^n K_i$ is an optimum for F

$$\text{iff } (0, 0, ..., 0) \in \partial(F + \chi_{\prod_{i=1}^n K_i})(\bar{x}_1, ..., \bar{x}_n),$$

$$\text{iff } \partial F(\bar{x}_1, ..., \bar{x}_n) \cap -\partial\chi_{\prod_{i=1}^n K_i}(\bar{x}_1, ..., \bar{x}_n) \neq \varnothing$$

(under hypothesis (H_1) or (H_1'). It suffices therefore to prove that

$$\partial\chi_{\prod_{i=1}^n K_i}(\bar{x}_1, ..., \bar{x}_n) = \prod_{i=1}^n \partial\chi_{K_i}(\bar{x}_i). \tag{2.1}$$

Indeed,

$$(y_1, \dots, y_n) \in \partial\chi_{\prod_{i=1}^n K_i}(\bar{x}_1, \dots, \bar{x}_n)$$

$$\text{iff } \max_{\substack{x_i \in K_i \\ i=1,2,\dots,n}} \operatorname{Re}(\langle x_1, y_1\rangle_1 + \cdots + \langle x_n, y_n\rangle_n)$$

$$= \operatorname{Re}\langle \bar{x}_1, y_1\rangle_1 + \cdots + \operatorname{Re}\langle \bar{x}_n, y_n\rangle_n,$$

$$\text{iff } \max_{x_i \in K_i} \operatorname{Re}\langle x_i, y_i\rangle_i = \operatorname{Re}\langle \bar{x}_i, y_i\rangle_i, \qquad i = 1, 2, \dots, n,$$

$$\text{iff } (y_1, \dots, y_n) \in \prod_{i=1}^n \partial\chi_{K_i}(\bar{x}_i).$$

Theorem 2.1 is a slight extension of a theorem of Pšeničniĭ and Rockafellar for convex programs (see, e.g., [3, p. 30]).

THEOREM 2.2. *Suppose* $F \in \operatorname{conv}(\prod_{i=1}^n X_i)$ *and that it is finite and continuous at* $(\bar{x}_1, \dots, \bar{x}_n) \in \prod_{i=1}^n K_i$. *Then in order that* $(\bar{x}_1, \dots, \bar{x}_n)$ *being a multioptimum for F imply that it is an optimum for F, it is sufficient that the following equality hold for the subdifferentials*:

$$\partial F(\bar{x}_1, \dots, \bar{x}_n) = \prod_{i=1}^n \partial F_{\bar{x}_1,\dots,\bar{x}_{i-1},\bar{x}_{i+1},\dots,\bar{x}_n}(\bar{x}_i). \tag{2.2}$$

Proof. We first note the following easily established inclusion for the subdifferentials:

$$\partial F(\bar{x}_1, \dots, \bar{x}_n) \subset \prod_{i=1}^n \partial F_{\bar{x}_1,\dots,\bar{x}_{i-1},\bar{x}_{i+1},\dots,\bar{x}_n}(\bar{x}_i). \tag{2.3}$$

Assume now that equality (2.2) holds in the above inclusion. In view of (1.4) we have that $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ is a multioptimum for F

$$\text{iff } 0 \in \partial(F_{\bar{x}_1,\dots,\bar{x}_{i-1},\bar{x}_{i+1},\dots,\bar{x}_n} + \chi_{K_i})(\bar{x}_i)$$

$$= \partial F_{\bar{x}_1,\dots,\bar{x}_{i-1},\bar{x}_{i+1},\dots,\bar{x}_n}(\bar{x}_i) + \partial\chi_{K_i}(\bar{x}_i), \qquad i = 1, 2, \dots, n,$$

$$\text{iff } \prod_{i=1}^n \partial F_{\bar{x}_1,\dots,\bar{x}_{i-1},\bar{x}_{i+1},\dots,\bar{x}_n}(\bar{x}_i) \cap \prod_{i=1}^n \partial\chi_{-K_i}(-\bar{x}_i) \neq \varnothing.$$

Employing (2.1) and (2.2), the last condition holds

$$\text{iff } \partial F(\bar{x}_1, \dots, \bar{x}_n) \cap -\partial\chi_{\prod_{i=1}^n K_i}(\bar{x}_1, \dots, \bar{x}_n) \neq \varnothing,$$

iff $(\bar{x}_1, \dots, \bar{x}_n)$ is an optimum for F. ∎

Remarks. (1) Let $F \in \operatorname{conv}(\prod_{i=1}^{n} X_i)$ and let it be finite and continuous at $(\bar{x}_1, ..., \bar{x}_n)$. Then the equality (2.2) holds for the subdifferentials if and only if the following equality holds for the directional derivatives:

$$\begin{aligned} F'(\bar{x}_1, ..., \bar{x}_n ; x_1, ..., x_n) &= \sum_{i=1}^{n} F'_{\bar{x}_1, ..., \bar{x}_{i-1}, \bar{x}_{i+1}, ..., \bar{x}_n}(\bar{x}_i ; x_i) \\ &= \sum_{i=1}^{n} F'(\bar{x}_1, ..., \bar{x}_n ; 0, 0, ..., x_i, ..., 0). \end{aligned} \tag{2.4}$$

In fact,

$$\begin{aligned} F'(\bar{x}_1, ..., \bar{x}_n ; x_1, ..., x_n) &= \max_{(y_1, ..., y_n) \in \partial F(\bar{x}_1, ..., \bar{x}_n)} \left\{ \sum_{i=1}^{n} \operatorname{Re}\langle x_i, y_i \rangle \right\} \\ &\leqslant \sum_{i=1}^{n} \max_{y_i \in \partial F_{\bar{x}_1, ..., \bar{x}_{i-1}, \bar{x}_{i+1}, ..., \bar{x}_n}(\bar{x}_i)} \operatorname{Re}\langle x_i, y_i \rangle \\ &= \sum_{i=1}^{n} F'_{\bar{x}_1, ..., \bar{x}_{i-1}, \bar{x}_{i+1}, ..., \bar{x}_n}(\bar{x}_i ; x_i). \end{aligned}$$

Hence, equality in the inclusion (2.3) for subdifferentials enforces equality in the above inequality. Conversely, suppose (2.4) holds and let

$$(y_1, y_2, ..., y_n) \in \prod_{i=1}^{n} \partial F_{\bar{x}_1, ..., \bar{x}_{i-1}, \bar{x}_{i+1}, ..., \bar{x}_n}(\bar{x}_i).$$

Then

$$\begin{aligned} \sum_{i=1}^{n} \operatorname{Re}\langle x_i, y_i \rangle &\leqslant \sum_{i=1}^{n} F'_{\bar{x}_1, ..., \bar{x}_{i-1}, \bar{x}_{i+1}, ..., \bar{x}_n}(\bar{x}_i ; x_i) = F'(\bar{x}_1, ..., \bar{x}_n ; x_1, ..., x_n) \\ &\leqslant F(x_1 + \bar{x}_1, ..., x_n + \bar{x}_n) - F(\bar{x}_1, ..., \bar{x}_n), \\ & \qquad x_i \in X_i, \ i = 1, 2, ..., n. \end{aligned}$$

Thus $(y_1, y_2, ..., y_n) \in \partial F(\bar{x}_1, ..., \bar{x}_n)$ and (2.2) holds.

(2) Again, let $F \in \operatorname{conv}(\prod_{i=1}^{n} X_i)$ and let it be finite and continuous at $(\bar{x}_1, ..., \bar{x}_n)$. Then F is Gâteaux-differentiable at $(\bar{x}_1, ..., \bar{x}_n)$ if and only if $F_{\bar{x}_1, ..., \bar{x}_{i-1}, \bar{x}_{i+1}, ..., \bar{x}_n}$ is Gâteaux-differentiable at $\bar{x}_i$, $i = 1, 2, ..., n$. In this case (2.2) evidently holds for the subdifferentials.

(3) Apart from the differentiable case of the preceding remark, another simple case, wherein (2.2) holds for the subdifferentials, is the following: $F(x_1, ..., x_n) = f_1(x_1) + \cdots + f_n(x_n)$, where $f_i \in \operatorname{conv}(X_i)$, $i = 1, 2, ..., n$.

(4) Condition (2.2) is not necessary in order for a multioptimum to be an optimum. To illustrate this, let $X_i = X$ be a Banach space, $i = 1, 2,..., n$, and let $F(x_1 ,..., x_n) = \| \sum_{i=1}^{n} x_i \|$. Let $(\bar{x}_1 ,..., \bar{x}_n) = (0, 0,..., 0)$. Then for arbitrarily given convex sets $K_i \subset X_i$ such that $(0,...,0) \in \prod_{i=1}^{n} K_i$, $(0, 0,..., 0)$ is a multioptimum implies that it is an optimum for F. However, in this case it is easily verified that (2.2) does not hold.

THEOREM 2.3. *Let $F \in \operatorname{conv}(\prod_{i=1}^{n} X_i)$ and let it be finite and continuous at $(\bar{x}_1 ,..., \bar{x}_n)$. Furthermore, suppose that $(0,..., 0) \notin \partial F(\bar{x}_1 ,..., \bar{x}_n)$ and that the following holds*:

$$\underbrace{\left(\prod_{i=1}^{n} \bigcup_{\lambda_i > 0} \lambda_i \, \partial F_{\bar{x}_1,\dots,\bar{x}_{i-1},\bar{x}_{i+1},\dots,\bar{x}_n}(\bar{x}_i)\right)}_{(\lambda_1 ,..., \lambda_n) \neq (1,..., 1)} \cap \partial F(\bar{x}_1 ,..., \bar{x}_n) = \varnothing. \tag{2.5}$$

Then in order that for arbitrarily given convex sets $K_i \subset X_i$ such that $\bar{x}_i \in K_i$, $i = 1, 2,..., n$, $(\bar{x}_1 ,..., \bar{x}_n)$ being a multioptimum for F imply that it is an optimum for F, it is necessary and sufficient that (2.2) *hold.*

Proof. The sufficiency part is already contained in Theorem 2.2. In order to prove the necessity part, suppose that (2.2) does not hold and let

$$(\bar{y}_1 ,..., \bar{y}_n) \in \left(\prod_{i=1}^{n} \partial F_{\bar{x}_1,\dots,\bar{x}_{i-1},\bar{x}_{i+1},\dots,\bar{x}_n}(\bar{x}_i)\right) \backslash \partial F(\bar{x}_1 ,..., \bar{x}_n). \tag{2.6}$$

Define the convex sets K_i as follows:

$$K_i = \{x_i \in X_i / \operatorname{Re}\langle x_i - \bar{x}_i , \bar{y}_i\rangle \geqslant 0\}, \qquad i = 1, 2,..., n.$$

Then by Theorem 2.1 one has that $(\bar{x}_1 ,..., \bar{x}_n)$ is a multioptimum for F on $\prod K_i$. To complete the proof of the theorem, we assert that $(\bar{x}_1 ,..., \bar{x}_n)$ is not an optimum for F on $\prod K_i$.

Assume the contrary. Then using Theorem 2.1 once more there exists an element

$$(\tilde{y}_1 ,..., \tilde{y}_n) \in \partial F(\bar{x}_1 ,..., \bar{x}_n) \cap \prod_{i=1}^{n} \partial \chi_{-K_i}(-\bar{x}_i).$$

Now let

$$H_i = \{x_i \in X_i / \operatorname{Re}\langle x_i - \bar{x}_i , \tilde{y}_i\rangle \geqslant 0\}, \qquad i = 1, 2,..., n.$$

Since $\tilde{y}_i \in \partial \chi_{-K_i}(-\bar{x}_i)$, one has $K_i \subset H_i$, $i = 1,..., n$. This inclusion of the half-spaces in X_i entails that $\tilde{y}_i = \lambda_i \bar{y}_i$, $\lambda_i > 0$, $i = 1, 2,..., n$, where not all the λ_i's are equal to 1 on account of (2.6). This contradicts (2.5) and establishes the theorem. ∎

3. Characterization of Optimum and Multioptimum in Question 1

Lemma 3.1. *Let* $f \in \operatorname{conv}(X)$ *and define* $F \in \operatorname{conv}(X \times X)$ *by* $F(x_1, x_2) = f(x_1 - x_2)(x_1, x_2 \in X)$. *Given* $\bar{x}_1, \bar{x}_2 \in X$, *one has*

$$\partial F_{\bar{x}_2}(\bar{x}_1) = \partial f(\bar{x}_1 - \bar{x}_2), \qquad \partial F_{\bar{x}_1}(\bar{x}_2) = -\partial f(\bar{x}_1 - \bar{x}_2) \tag{3.1}$$

and

$$\partial F(\bar{x}_1, \bar{x}_2) = \{(y_1, y_2)/y_2 = -y_1, y_1 \in \partial f(\bar{x}_1 - \bar{x}_2)\}. \tag{3.2}$$

Proof. Relations (3.1) and the inclusion

$$\{(y_1, y_2)/y_2 = -y_1, y_1 \in \partial f(\bar{x}_1 - \bar{x}_2)\} \subset \partial F(\bar{x}_1, \bar{x}_2)$$

are obvious.

In order to reverse this inclusion, suppose $(y_1, y_2) \in \partial F(\bar{x}_1, \bar{x}_2)$. Then in view of (2.3) and (3.1) one has $(y_1, y_2) \in \partial f(\bar{x}_1 - \bar{x}_2) \times -\partial f(\bar{x}_1 - \bar{x}_2)$. Taking into account the definition of subdifferential the inequality

$$f(x_1 - x_2) \geqslant f(\bar{x}_1 - \bar{x}_2) + \operatorname{Re}\langle x_1 - \bar{x}_1, y_1\rangle + \operatorname{Re}\langle x_2 - \bar{x}_2, y_2\rangle$$

holds for all $x_1, x_2 \in X$. Hence, in particular it holds for $x_1, x_2 \in X$ satisfying $x_1 - \bar{x}_1 = x_2 - \bar{x}_2$. Thus for all $x \in X$ we have $\operatorname{Re}\langle x, y_1 + y_2\rangle \leqslant 0$, which yields $y_1 + y_2 = 0$.

Employing Lemma 3.1 and Theorem 2.1, one immediately obtains

Theorem 3.2. *Let* $f \in \operatorname{conv}(X)$ *and let* U, V *be convex sets in* X. *Assume that either*

(H_2) $\operatorname{dom}(f) \cap \operatorname{int}(U - V) \neq \varnothing$

or

(H_2') f *is* d-*continuous and* $\operatorname{int}\operatorname{dom}(f) \cap (U - V) \neq \varnothing$

holds. Then $(\bar{u}, \bar{v}) \in U \times V$ *is an optimum for* f *if and only if there exists an element* $y \in Y$ *such that*

(i) $y \in \partial f(\bar{u} - \bar{v})$,

(ii) $\operatorname{Re}\langle \bar{u}, y\rangle = \inf_{u \in U} \operatorname{Re}\langle u, y\rangle$,

(iii) $\operatorname{Re}\langle \bar{v}, y\rangle = \sup_{v \in V} \operatorname{Re}\langle v, y\rangle$.

The next theorem furnishes an answer to Question 1 as a particular case of Theorem 2.2.

Theorem 3.3. *Let* $f \in \operatorname{conv}(X)$ *and let it be finite and continuous at* $\bar{u} - \bar{v}$. *Then in order that* $(\bar{u}, \bar{v}) \in U \times V$ *being a multioptimum for* f *imply that it is an optimum for* f, *it is sufficient that* f *be Gâteaux-differentiable at* $\bar{u} - \bar{v}$.

Proof. Due to the assumption that f is finite and continuous at $\bar{u} - \bar{v}$, we note that $\partial f(\bar{u} - \bar{v})$ consists of a single element if and only if f is Gâteaux-differentiable at $\bar{u} - \bar{v}$. Moreover, in view of Lemma 3.1 and the above observation it follows that if we take $F(u, v) = f(u - v)$, then the equality $\partial F(\bar{u}, \bar{v}) = \partial F_{\bar{v}}(\bar{u}) \times \partial F_{\bar{u}}(\bar{v})$ holds for the subdifferentials if and only if f is Gâteaux-differentiable at $\bar{u} - \bar{v}$. The proof is completed by applying Theorem 2.2. ∎

Remark. Theorem 3.3 remains valid if instead of assuming that f is finite and contnuous at $\bar{u} - \bar{v}$, we make any one of the following weaker hypotheses:

(H_3) $(\bar{u} - \text{dom}(f)) \cap \text{int } V \neq \varnothing$ and $(\bar{v} + \text{dom}(f)) \cap \text{int } U \neq \varnothing$;

(H_3') f is d-continuous and $\text{int}(\bar{u} - \text{dom}(f)) \cap V \neq \varnothing$, $\text{int}(\bar{v} + \text{dom}(f)) \cap U \neq \varnothing$.

The assertion follows easily by taking into account the first observation in the proof of Theorem 3.3 and then employing Theorem 3.2.

COROLLARY 3.4. *Let $f \in \text{conv}(X)$ and let it be finite and continuous at $\bar{x} \in X$. Further suppose that $0 \notin \partial f(\bar{x})$ and that $\{\bigcup_{\lambda>0,\lambda\neq 1} \lambda \partial f(\bar{x})\} \cap \partial f(\bar{x}) = \varnothing$. Then in order that, for arbitrary given convex sets U, V such that $\bar{x} \in U$ and $0 \in V$, $(\bar{x}, 0)$ being a multioptimum for f imply that it is an optimum for f, it is necessary and sufficient that f be Gâteaux-differentiable at $\bar{x}$.*

Proof. This follows immediately from Theorem 2.3 and Lemma 3.1. ∎

Corollary 3.4 can be strengthened in the case where f is a gauge function on X, i.e., a real-valued function on X satisfying $f(x_1 + x_2) \leqslant f(x_1) + f(x_2)$ for all $x_1, x_2 \in X$ and $f(\lambda x) = \lambda f(x)$ for all $x \in X$ and $\lambda \geqslant 0$. In this case the subdifferential of f at $\bar{x}$ is given by $\partial f(\bar{x}) = \{y \in \partial f(\theta) / f(\bar{x}) = \text{Re}\langle \bar{x}, y\rangle\}$, where θ denotes the zero vector of X. One thus obtains

THEOREM 3.5. *Let f be a continuous gauge function defined on X. Then in order that for arbitrarily given convex sets U, V in X and points $\bar{u} \in U$, $\bar{v} \in V$ such that $f(\bar{u} - \bar{v}) \neq 0$, $(\bar{u}, \bar{v})$ being a multioptimum for f imply that it is an optimum for f, it is necessary and sufficient that f be Gâteaux-differentiable at each point $x \in X$, where $f(x) \neq 0$.*

Proof. The sufficiency part is already contained in Theorem 3.3. To prove the necessity part, suppose there exists a point $\bar{x} \in X$ such that $f(\bar{x}) \neq 0$ and such that f is not Gâteaux-differentiable at $\bar{x}$. Then there exist $y_1, y_2 \in \partial f(\theta)$, $y_1 \neq y_2$, such that $\text{Re}\langle \bar{x}, y_1\rangle = \text{Re}\langle \bar{x}, y_2\rangle = f(\bar{x})$. Let us assume first that $f(\bar{x}) > 0$. Now select $\tilde{x} \in X$ such that $0 < \text{Re}\langle \tilde{x}, y_1\rangle <$

$\text{Re}\langle\tilde{x}, y_2\rangle$ and let $U = \{x/\text{Re}\langle x, y_2\rangle \geqslant f(\bar{x})\}$, $V = \{x/\text{Re}\langle x, y_1\rangle = 0\}$. Then $\bar{x} \in U$, $0 \in V$ and $(\bar{x}, 0)$ is a multioptimum for f, but it is not an optimum for f.

In fact, let $\hat{x} = \tilde{x}f(\bar{x})/\text{Re}\langle\tilde{x}, y_2\rangle$. Then $\text{Re}\langle\hat{x}, y_1\rangle < f(\bar{x}) = \text{Re}\langle\hat{x}, y_2\rangle$. This gives $\hat{x} \in U$, $\hat{x} - \text{Re}\langle\hat{x}, y_1\rangle(\bar{x}/f(\bar{x})) \in V$ and

$$f\left(\hat{x} - \left(\hat{x} - \text{Re}\langle\hat{x}, y_1\rangle \frac{\bar{x}}{f(\bar{x})}\right)\right) = \text{Re}\langle\hat{x}, y_1\rangle < f(\bar{x}).$$

In case $f(\bar{x}) < 0$ we select $\tilde{x} \in X$ such that $0 < \text{Re}\langle\tilde{x}, y_2\rangle < \text{Re}\langle\tilde{x}, y_1\rangle$ and proceed exactly as before. ∎

In the last part of this section we consider the particular case when f is a seminorm defined as follows. Let B be a balanced and equicontinuous subset of Y and let

$$f(x) = \sup_{y \in B} \text{Re}\langle x, y\rangle, \qquad x \in X.$$

We note that the set $K = \overline{\text{co}}(B)$ is a balanced convex and $\sigma(Y, X)$-compact subset of Y and hence we have $f(x) = \max_{y \in K} \text{Re}\langle x, y\rangle$ and

$$\partial f(\bar{x}) = \{y \in K/f(\bar{x}) = \text{Re}\langle\bar{x}, y\rangle\}.$$

In the case of a real locally convex space X, the above seminorm f has been employed by Laurent [6, p. 426].

COROLLARY 3.6. *Suppose that K is not contained in any closed hyperplane, core*[2] *$(K) \neq \varnothing$ and that K is strictly convex, i.e., $x_1, x_2 \in K$, $0 < \lambda < 1$, imply* $(1 - \lambda) x_1 + \lambda x_2 \in \text{core}(K)$. *Then for arbitrarily given convex sets U, V in X and points $\bar{u} \in U$, $\bar{v} \in V$, $(\bar{u}, \bar{v})$ is a multioptimum for f implies that it is an optimum for f.*

Proof. In view of Theorem 3.5 it suffices to prove that for each $x \in X$, $x \neq \theta$, $\partial f(x)$ consists of a single element. We note that for $\theta \neq x \in X$ the set $\partial f(x)$ is a nonempty $\sigma(Y, X)$-compact and proper extremal subset of K. Hence, if $\partial(x)$ contained more than one point, say the points y_1 and y_2 with $y_1 \neq y_2$, then $(1 - \lambda) y_1 + \lambda y_2 \notin \text{core}(K)$, $0 < \lambda < 1$. This contradicts the strict convextity of K and establishes the corollary. ∎

4. Characterization of Optimum for the Case of Finite Dimensional Convex Sets

Here we consider the important special cases when the convex sets K_i of Section 1 and the convex sets U, V of Section 3 are contained in subspaces

[2] Recall that $\text{Core}(K) = \{k \in K/\forall_{k' \in Y}, \exists_{\epsilon > 0} \ni \forall_{\lambda \in [-\epsilon, +\epsilon]}, k + \lambda k' \in K\}$.

of finite dimension. For this purpose we adopt essentially the approach as given in [6, Theorem 8.3.3, p. 438]. However, with a suitable modification of arguments, it is shown that Theorem 8.3.3 of [6] holds for a convex set in place of a linear subspace and thereby extends to cover the cases of optima dealt with in Section 2 and Section 3.

Let $K \subset X$ be convex and let $\bar{x} \in K$. By

$$C(K; \bar{x}) = \overline{\bigcup_{\lambda>0} \lambda(K - \bar{x})}$$

we denote the support cone of K at $\bar{x}$. Let $L(K; \bar{x}) = C(K; \bar{x}) \cap -C(K; \bar{x})$ stand for the largest subspace contained in the support cone $C(K; \bar{x})$. By the facet of $\bar{x}$ in K we will mean the set $H(K; \bar{x}) = (\bar{x} + L(K; \bar{x})) \cap K$. Note that $H(K; \bar{x})$ is the smallest extremal subset of K containing $\bar{x}$. Hence, $\bar{x}$ is and extreme point of K if and only if $H(K; \bar{x}) = \{\bar{x}\}$ (or equivalently $L(K; \bar{x}) = \{\theta\}$).

LEMMA 4.1. *Let X be of finite dimension n. Suppose $f \in \operatorname{conv}(X)$ and let it be finite and continuous at $\bar{x} \in X$. Suppose $y \in \partial f(\bar{x})$. Then there exist m elements $y_i \in \operatorname{Ext}$[3] $(\partial f(\bar{x}))$, $i = 1, 2,..., m$, and m numbers $\lambda_i > 0$, $\sum_{i=1}^{m} \lambda_i = 1$, such that $y = \sum_{i=1}^{m} \lambda_i y_i$, with $1 \leqslant m \leqslant n + 1$ (for real scalars) or $1 \leqslant m \leqslant 2n + 1$ (for complex scalars).*

The lemma is well known (cf. [6, p. 436]). It is an immediate consequence of the Krein–Milman Theorem, a theorem of Carathéodory and the fact that $\partial f(\bar{x})$ is a nonempty $\sigma(Y, X)$-compact convex set.

THEOREM 4.2. *Let $F \in \operatorname{conv}(\prod_{i=1}^{n} X_i)$ and let it be finite and continuous at $(\bar{x}_1,..., \bar{x}_n) \in \prod_{i=1}^{n} K_i$, where $K_i \subset X_i$ are convex sets such that $\dim[K_i]$[4] $= m_i$, $i = 1,..., n$. Then $(\bar{x}_1,..., \bar{x}_n)$ is an optimum for F if and only if there exist s elements $(y_1^{(j)},..., y_n^{(j)}) \in \operatorname{Ext} \partial F(\bar{x}_1,..., \bar{x}_n)$, $1 \leqslant j \leqslant s$, and s numbers $\lambda_j > 0$, with $\sum_{j=1}^{s} \lambda_j = 1$ such that*

(i) *$1 \leqslant s \leqslant \sum_{i=1}^{n} m_i + 1$ (real scalars) or $1 \leqslant s \leqslant 2 \sum_{i=1}^{n} m_i + 1$ complex scalars),*

(ii) *$\operatorname{Re} \sum_{j=1}^{s} \lambda_j \langle \bar{x}_i - x_i, y_i^{(j)} \rangle_i \leqslant 0$ $(x_i \in K_i)$, $i = 1, 2,..., n$.*

Proof. The sufficiency part of the theorem is trivial. To prove the necessity part, let $(\bar{x}_1,..., \bar{x}_n) \in \prod_{i=1}^{n} K_i$ be an optimum for F. Then by Theorem 2.1 one has

$$\partial F(\bar{x}_1,..., \bar{x}_n) \cap \prod_{i=1}^{n} -\partial \chi_{K_i}(\bar{x}_i) \neq \varnothing. \tag{4.1}$$

[3] As usual, $\operatorname{Ext}(A)$ denotes the set of extreme points of A.

[4] We employ the notation $[K_i]$ for the span of K_i.

We note that with the $\sigma(Y_i, X_i)$ topology on Y_i, $i = 1, 2,..., n$, the topology of the product for $\prod_{i=1}^n Y_i$ coincides with the $\sigma(\prod_{i=1}^n Y_i, \prod_{i=1}^n X_i)$ topology with respect to the corresponding duality (cf. [5, Theorem 17.14, p. 160]). Thus the set $\prod_{i=1}^n - \partial\chi_{K_i}(\bar{x}_i)$ is $\sigma(\prod Y_i, \prod X_i)$-closed and this entails that the set on the left-hand side of (4.1) is $\sigma(\prod Y_i, \prod X_i)$-compact. By the Krein–Milman theorem we have $\operatorname{Ext}(\partial F(\bar{x}_1,..., \bar{x}_n) \cap \prod_{i=1}^n - \partial\chi_{K_i}(\bar{x}_i)) \neq \varnothing$. Now let

$$(\bar{y}_1,..., \bar{y}_n) \in \operatorname{Ext}\left(\partial F(\bar{x}_1,..., \bar{x}_n) \cap \prod_{i=1}^n -\partial\chi_{K_i}(\bar{x}_i)\right).$$

Then we have

$$\begin{aligned} \{(\theta,..., \theta)\} &= L\left(\partial F(\bar{x}_1,..., \bar{x}_n) \cap \prod_{i=1}^n -\partial\chi_{K_i}(\bar{x}_i); (\bar{y}_1,..., \bar{y}_n)\right) \qquad (4.2) \\ &= L(\partial F(\bar{x}_1,..., \bar{x}_n); (\bar{y}_1,..., \bar{y}_n)) \cap L\left(\prod_{i=1}^n -\partial\chi_{K_i}(\bar{x}_i); (\bar{y}_1,..., \bar{y}_n)\right) \end{aligned}$$

(cf. [6, Proposition 8.3.2]).

On the other hand,

$$\left(\prod_{i=1}^n [K_i]\right)^\perp \subset C\left(\prod_{i=1}^n -\partial\chi_{K_i}(\bar{x}_i); (\bar{y}_1,..., \bar{y}_n)\right),$$

where $A^\perp$ stands for the annihilator subspace. For if we assume $(y_1,..., y_n) \in (\prod_{i=1}^n [K_i])^\perp$, then clearly

$$(y_1 + \bar{y}_1,..., y_n + \bar{y}_n) \in \prod_{i=1}^n -\partial\chi_{K_i}(\bar{x}_i).$$

Hence, we get that

$$\left(\prod_{i=1}^n [K_i]\right)^\perp \subset L\left(\prod_{i=1}^n -\partial\chi_{K_i}(\bar{x}_i); (\bar{y}_1,..., \bar{y}_2)\right).$$

Since

$$L\left(\prod_{i=1}^n -\partial\chi_{K_i}(\bar{x}_i); (\bar{y}_1,..., \bar{y}_n)\right)$$

is the largest subspace contained in the cone

$$C\left(\prod_{i=1}^n -\partial\chi_{K_i}(\bar{x}_i); (\bar{y}_1,..., \bar{y}_n)\right).$$

As a consequence, one has that

$$\operatorname{codim} L\left(\prod_{i=1}^{n} -\partial\chi_{K_i}(\bar{x}_i); (\bar{y}_1, ..., \bar{y}_n)\right) \leqslant \dim\left(\prod_{i=1}^{n} [K_i]\right) = \sum_{i=1}^{n} m_i . \quad (4.3)$$

From (4.2) and (4.3) we conclude that $L(\partial F(\bar{x}_1, ..., \bar{x}_n); (\bar{y}_1, ..., \bar{y}_n))$ is a subspace of dimension at most equal to $\sum_{i=1}^{n} m_i$. The remaining argument is exactly the same as that given in [6, Theorem 8.3.3]. In fact, $H(\partial F(\bar{x}_1, ..., \bar{x}_n); (\bar{y}_1, ..., \bar{y}_n))$ is a $\sigma(\prod Y_i, \prod X_i)$-compact convex set contained in a linear variety of dimension at most equal to $\sum_{i=1}^{n} m_i$ ($2\sum_{i=1}^{n} m_i$ for the complex case). Hence, to conclude the proof it only remains to apply Lemma 4.1 and to employ the fact that $H(\partial F(\bar{x}_1, ..., \bar{x}_n); (\bar{y}_1, ..., \bar{y}_n))$ is an extremal subset of $\partial F(\bar{x}_1, ..., \bar{x}_n)$. ∎

COROLLARY 4.3. *Let $f \in \operatorname{conv}(X)$ and let it be finite and continuous at $\bar{u} - \bar{v}$. Suppose $\bar{u} \in U$ and $\bar{v} \in V$, where U, V are convex sets such that* $\dim[U] = m$ *and* $\dim[V] = n$. *Then $(\bar{u}, \bar{v})$ is an optimum for f if and only if there exist k elements $y_i \in \operatorname{Ext} \partial f(\bar{u} - \bar{v})$ and k numbers $\lambda_i > 0$ with $\sum_{i=1}^{k} \lambda_i = 1$ such that*

(i) $1 \leqslant k \leqslant m + n + 1$, *(real scalars)*, $1 \leqslant k \leqslant 2m + 2n + 1$, *(complex scalars)*,

(ii) $\operatorname{Re} \sum_{i=1}^{k} \lambda_i \langle \bar{u} - u, y_i \rangle \leqslant 0 \ (u \in U)$,

(iii) $\operatorname{Re} \sum_{i=1}^{k} \lambda_i \langle \bar{v} - v, y_i \rangle \geqslant 0 \ (v \in V)$.

Proof. This follows immediately upon applying Lemma 3.1 and Theorem 4.2, ∎

Theorem 4.2 can be generalized so as to be valid under the slightly weaker hypothesis (H_1'). For this purpose we again adopt basically the same approach as that given in [6, Theorem 8.3.6].

We recall that an extremal ray D of a set $A \subset X$ is a closed semiline contained in A, which is also an extremal subset of A. Extreme directions of A are elements d such that A contains an extremal ray of the type $D = \{x/x = x_0 + \lambda d, \lambda \geqslant 0\}$.

THEOREM 4.4. *Let $F \in \operatorname{conv}(\prod_{i=1}^{n} X_i)$ and let it satisfy* (H_1'). *Let $K_i \subset X_i$ be convex sets such that* $\dim[K_i] = m_i$, $i = 1, ..., n$. *Then $(\bar{x}_1, ..., \bar{x}_n) \in \prod_{i=1}^{n} K_i$ is an optimum for F if and only if there exist s elements $(y_1^{(j)}, ..., y_n^{(j)}) \in$* $\operatorname{Ext} \partial F(\bar{x}_1, ..., \bar{x}_n)$, $1 \leqslant j \leqslant s$ *and t elements $(d_1^{(j)}, ..., d_n^{(j)})$, $1 \leqslant j \leqslant t$, $t \geqslant 0$, that are extreme directions of $\partial F(\bar{x}_1, ..., \bar{x}_n)$, with $1 \leqslant s + t \leqslant \sum_{i=1}^{n} m_i + 1$*

($2\sum_{i=1}^{n} m_i + 1$ *for complex scalars*) *and positive numbers* $\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_t$, $\sum_{j=1}^{s} \lambda_j = 1$, *such that*

$$\operatorname{Re}\left\{\sum_{j=1}^{s} \lambda_j \langle \bar{x}_i - x_i, y_j^{(i)} \rangle_i + \sum_{j=1}^{t} \mu_j \langle \bar{x}_i - x_i, d_i^{(j)} \rangle_i \right\} \leqslant 0$$

$$(x_i \in K_i), \quad i = 1, 2, \ldots, n.$$

Proof. The sufficiency part of the theorem clearly follows from Theorem 2.1. To prove the necessity part, let $(\bar{x}_1, \ldots, \bar{x}_n) \in \prod_{i=1}^{n} K_i$ be an optimum for F. Since F is d-continuous the set $\partial F(\bar{x}_1, \ldots, \bar{x}_n)$ is a $\sigma(\prod_{i=1}^{n} Y_i, \prod_{i=1}^{n} X_i)$-closed convex locally compact set not containing a line (cf. [4]). Employing Theorem 2.1 and Theorem 8.3.6(i) of Laurent [6, p. 441], there exists an element

$$(\bar{y}_1, \ldots, \bar{y}_n) \in \operatorname{Ext}\left(\partial F(\bar{x}_1, \ldots, \bar{x}_n) \cap \prod_{i=1}^{n} -\partial \chi_{K_i}(\bar{x}_i)\right).$$

Proceeding exactly as in the proof of Theorem 4.2, we obtain that $H(\partial F(\bar{x}_1, \ldots, \bar{x}_n); (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n))$ is a $\sigma(\prod_{i=1}^{n} Y_i, \prod_{i=1}^{n} X_i)$-closed convex set not containing a line and such that it is contained in a linear variety of dimension equal to $\sum_{i=1}^{n} m_i$ ($2\sum_{i=1}^{n} m_i$ for the complex case). The remaining argument is the same as that given in [6, Theorem 8.3.7]. ∎

COROLLARY 4.5. *Let* $f \in \operatorname{conv}(X)$ *and let it satisfy* (H_2'). *Let* U, V *be convex sets such that* $\dim[U] = m$, $\dim[V] = n$, *and let* $\bar{u} \in U$, $\bar{v} \in V$. *Then* $(\bar{u}, \bar{v})$ *is an optimum for* f *if and only if there exist* k *elements* y_i, $i = 1, \ldots, k$, $k \geqslant 1$, *that are extreme points of* $\partial f(\bar{u} - \bar{v})$ *and* s *elements* d_i, $i = 1, \ldots, s$, $s \geqslant 0$, *that are extreme directions of* $\partial f(\bar{u} - \bar{v})$ *with* $1 \leqslant k + s \leqslant m + n + 1$ ($2m + 2n + 1$ *for the complex case*) *and positive numbers* $\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_s$, *such that*

(i) $\operatorname{Re}\{\sum_{i=1}^{k} \lambda_i \langle \bar{u} - u, y_i \rangle + \sum_{i=1}^{s} \mu_i \langle \bar{u} - u, d_i \rangle\} \leqslant 0$ $(u \in U)$,

(ii) $\operatorname{Re}\{\sum_{i=1}^{k} \lambda_i \langle \bar{v} - v, y_i \rangle + \sum_{i=1}^{s} \mu_i \langle \bar{v} - v, d_i \rangle\} \geqslant 0$ $(v \in V)$.

5. APPLICATIONS

Here we consider two specific examples, wherein the results of the previous sections are applicable.

a. *Multivariate Constrained Convex Optimization*

Let $F \in \operatorname{conv}(\prod_{i=1}^{n} X_i)$ and let it be finite everywhere. Let $f_i^{(j)} \in \operatorname{conv}(X_i)$ and let it be continuous on X_i, $j = 1, 2,..., m_i$, $i = 1, 2,..., n$. Furthermore, let the convex sets $K_i \subset X_i$ be defined as follows:

$$K_i = K_i^{(1)} \cap K_i^{(2)} \cap \cdots \cap K_i^{(m_i)}, \qquad i = 1, 2,..., n,$$

where

$$K_i^{(j)} = \{x_i \in X_i / f_i^{(j)}(x_i) \leqslant 0\}, \qquad j = 1, 2,..., m_i \text{ and } i = 1, 2,..., n,$$

In addition, we make the following regularity hypothesis on the functions $f_i^{(j)}$:

$$(\mathrm{R}_1) \qquad \bigcap_{j=1}^{m_i} \mathring{K}_i^{(j)} \neq \varnothing, \qquad i = 1, 2,..., n.$$

Then Theorem 2.1 takes the following particular form.

THEOREM 5.1. (Kuhn–Tucker-type characterization) *If the hypothesis* (R_1) *is fulfilled, then* $(\bar{x}_1,..., \bar{x}_n) \in \prod_{i=1}^{n} K_i$ *is an optimum for* F *if and only if there exist elements* $y_i^{(j)} \in \partial f_i^{(j)}(x_i)$ *and numbers* $\lambda_i^{(j)} \leqslant 0$, $j = 1, 2,..., m_i$, $i = 1, 2,..., n$, *such that* $\lambda_i^{(j)} f_i^{(j)}(\bar{x}_i) = 0, j = 1,..., m_i$, $i = 1, 2,..., n$, *and*

$$\left(\sum_{j=1}^{m_1} \lambda_1^{(j)} y_1^{(j)},..., \sum_{j=1}^{m_n} \lambda_n^{(j)} y_n^{(j)}\right) \in \partial F(\bar{x}_1,..., \bar{x}_n).$$

Proof. We have $\chi_{K_i} = \sum_{j=1}^{m_i} \chi_{K_i^{(j)}}$ and in view of the hypothesis (R_1) one has

$$\partial \chi_{K_i}(\bar{x}_i) = \sum_{j=1}^{m_i} \partial \chi_{K_i^{(j)}}(\bar{x}_i).$$

Furthermore, we note that in this case the subdifferential $\partial \chi_{K_i^{(j)}}(\bar{x}_i)$ has the following explicit expression:

$$\begin{aligned} \partial \chi_{K_i^{(j)}}(\bar{x}_i) &= \varnothing, && \text{if } f_i^{(j)}(\bar{x}_i) > 0, \\ &= \{\theta\}, && \text{if } f_i^{(j)}(\bar{x}_i) < 0, \\ &= -C(\theta, \partial f_i^{(j)}(\bar{x}_i)), && \text{if } f_i^{(j)}(\bar{x}_i) = 0, \end{aligned}$$

(cf. [3, p. 32].

To complete the proof it suffices to observe that by Theorem 2.1, $(\bar{x}_1, ..., \bar{x}_n) \in \prod_{i=1}^n K_i$ is an optimum for F if and only if

$$\partial F(\bar{x}_1, ..., \bar{x}_n) \cap \prod_{i=1}^n -\partial\chi_{K_i}(\bar{x}_i) \neq \varnothing. \quad \blacksquare$$

COROLLARY 5.2. *Let $f \in \text{conv}(X)$ and let it be finite everywhere. Let $g_i \in \text{conv}(X)$ and let it be continuous, $i = 1, 2, ..., l$. Let $h_j \in \text{conv}(X)$ and let it be continuous, $j = 1, 2, ..., m$. Let the convex sets U, V be defined by $U = \bigcap_{i=1}^l U_i$, $V = \bigcap_{j=1}^m V_j$. Furthermore, suppose that the following regularity hypothesis is satisfied:*

$$(\mathrm{R}_2) \qquad \bigcap_{i=1}^l \mathring{U}_i \neq \varnothing \quad \text{and} \quad \bigcap_{j=1}^m \mathring{V}_j \neq \varnothing.$$

Then $(\bar{u}, \bar{v})$ is an optimum for f if and only if there exist

$$y_i \in \partial g_i(\bar{u} - \bar{v}), \qquad \lambda_i \leqslant 0, \quad i = 1, 2, ..., l,$$

and

$$y_j' \in \partial h_j(\bar{u} - \bar{v}), \qquad \lambda_j' \geqslant 0, \quad j = 1, 2, ..., m,$$

such that

$$\lambda_i g_i(\bar{u} - \bar{v}) = 0, \qquad i = 1, 2, ..., l, \quad \lambda_j' h_j(\bar{u} - \bar{v}) = 0, \quad j = 1, 2, ..., m,$$

and

$$\sum_{i=1}^l \lambda_i y_i = \sum_{j=1}^m \lambda_j' y_j' \in \partial f(\bar{u} - \bar{v}).$$

Proof. This follows immediately from Theorem 5.1 upon applying Lemma 3.1. $\blacksquare$

b. *Global Simultaneous Approximation*

Let X be a normed linear space and let $K_i \subset X$, $i = 1, 2, ..., n$, be convex sets. For $1 \leqslant p \leqslant \infty$ we consider the following optimization problems:

(Pb_p) Minimize $\{\|x_1 - x_2\|^p + \|x_1 - x_3\|^p + \cdots + \|x_1 - x_n\|^p\}^{1/p}$ for $x_i \in K_i$, $i = 1, 2, ..., n$, where $1 \leqslant p < \infty$.

(Pb_∞) Minimize $\{\max(\|x_1 - x_2\|, \|x_1 - x_3\|, ..., \|x_1 - x_n\|)\}$ for $x_i \in K_i$, $i = 1, 2, ..., n$.

For the case when $n = 2$ these problems coincide with the problem of determining proximal points of convex sets which has been dealt with in [10]. On the other hand, when each one of the sets $K_2, ..., K_n$ is reduced to a singleton set these problems coincide with the so-called l_p-problems of simultaneous approximation. In case $X = \mathscr{C}[a, b]$, the space of continuous functions with the uniform norm, and K_1 is taken to be a unisolvent family of degree n, the l_∞-problem of simultaneous approximation has been con-considered in [2]. A more general problem of global approximation of a compact set has been treated in [7]. Here we particularize Theorem 2.1 so as to obtain a characterization of solutions to the above problems.

THEOREM 5.3. *Let $p' = p/(p-1)$ if $1 < p < \infty$, $p' = \infty$ if $p = 1$ and $p' = 1$ if $p = \infty$. Then in order that $(\bar{x}_1, ..., \bar{x}_n) \in \prod_{i=1}^n K_i$ be a solution to the problem* (Pb$_p$), $1 \leqslant p \leqslant \infty$, *it is necessary and sufficient that there exist $y_i \in S(X^*)$, $S(X^*)$ being the unit sphere of X^*, $i = 1, 2, ..., n$, such that*

(i) $\sum_{i=1}^n y_i = 0$,

(ii) $(\sum_{i=2}^n \| y_i \|^{p'})^{1/p'} = 1$ (for $p' = \infty$, $\max_{2 \leqslant i \leqslant n} \| y_i \| = 1$),

(iii) $\mathrm{Re}\langle \bar{x}_i - x_i, y_i \rangle \leqslant 0$ $(x_i \in K_i)$, $i = 1, 2, ..., n$,

(iv) $\mathrm{Re} \sum_{i=1}^n \langle \bar{x}_i, y_i \rangle = (\sum_{i=2}^n \| \bar{x}_1 - \bar{x}_i \|^p)^{1/p}$ (for $p = \infty$, $\max_{2 \leqslant i \leqslant n} \| \bar{x}_1 - \bar{x}_i \|$).

Proof. We set

$$F(x_1, ..., x_n) = \left(\sum_{i=2}^n \| x_1 - x_i \|^p \right)^{1/p}, \qquad 1 \leqslant p < \infty,$$

$$= \max_{2 \leqslant i \leqslant n} \{ \| x_1 - x_i \| \}, \qquad p = \infty,$$

and note that F is a gauge function on X^n. It is then easily verified that $\partial F(\bar{x}_1, ..., \bar{x}_n)$ is given by

$$\{(y_1, ..., y_n) / y_i \in S(X^*), \quad i = 1, 2, ..., n \text{ and } y_i \text{ satisfying (i), (ii) and (iv)}\}.$$

The proof is completed by applying Theorem 2.1. ∎

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